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Rational singularities on rational surfaces

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§0. Introduction:

We attempt to classify certain two dimensional local rings. This has been made possible by the recent developments due to Iitaka, Miyanishi, Sugie and Fujita.

We would like to study when are two local rings A and B isomorphic. Stated in this generality, the problem is hopeless. In algebraic geometry, in general the analytic local rings are easier to study. So let us further assume that $\hat{A} \simeq \hat{B}$, where $\hat{}$ denotes the completion. Looking at the case of regular local rings, another obvious assumption to make is that quotient fields of A and B are isomorphic. Even this hypothesis is not sufficient can be seen by looking at genus of functions at points of a curve of genus bigger than one. So we make a further assumption that these points lie on rational varieties.

Now let me state the problem. We consider normal two dimensional local rings, which are genus of functions on a rational surface over an algebraically closed field k of characteristic zero. If A and B are any two such rings with $\hat{A} \simeq \hat{B}$, is $A \simeq B$? When A is regular, this is a well known theorem of Nagata [N]. So, here we will be interested in certain normal local rings.

§1. Logarithmic Kodaira dimension of local rings:

Let A be any two dimensional normal local ring as above. Then we may find a projective surface X and $x \in X$ such that, $X - \{x\}$ is smooth and $\mathcal{O}_{X,x} \simeq A$.

Def: logarithmic Kodaira dimension $\bar{\kappa}(A)$ is defined as logarithmic Kodaira dimension of $X - \{x\}$.

It is easy to see that the definition depends only on A. Now we have one more invariant of local rings. We will use this invariant for the classification and it is very similar to the classification of surfaces.

In this section we will consider local rings A such that $\hat{A} \simeq K[[X^n, X^{n-1}Y,\dots,Y^n]]$. These are in some sense very simple singular local rings. If $X \to \operatorname{Spec} A$ is the minimal desingularisation, then the exceptional fibre is just one non-singular rational curve with self intersection number $-n$. So the problem becomes that of classifying embeddings of $\mathbb{P}'$ in a non-singular projective rational surface, upto certain obvious equivalences.

The theorem proved is the following:

Theorem 1: Let X be a complete nonsingular rational surface, $C \simeq \mathbb{P}'$, $C \subset X$ with $C^2 = -n$. $\bar{\kappa}$ as usual denotes logarithmic Kodaira dimension.

Then,

1) $\bar{\kappa}(X-C) = -\infty \Leftrightarrow$ There exists a birational map $f: X \to \mathbb{F}_n$ (Hirzebruch surface) which is an isomorphism in the

neighbourhood C and that of the unique negative curve in $\mathbb{F}_n$.

2) $\bar{\kappa}(X-C) = 0 \Leftrightarrow$ There exists a birational morphism $f: X \to \mathbb{P}^2$ and $f(C)$ is a sextic with ten double points.

3) $\bar{\kappa}(X-C) = 1 \Leftrightarrow$ There exists a birational morphism $f: X \to \mathbb{P}^2$ and $\deg f(C) = 3m$, $m > 2$ with 9 m-tuple points and one double point.

Thus we see that, if $\bar{\kappa}(A) = -\infty$ and $\hat{A} \simeq K[[X^n, X^{n-1}Y,\dots,Y^n]]$, then $A \simeq K[X^n, X^{n-1}Y,\dots,Y^n]$ localised at the obvious maximal ideal. In the other cases, no such uniqueness statement can be made.

This gives another proof of a Theorem of Enriques, which states that a curve in $\mathbb{P}^2$ can be transformed to a straight line by a Cremona transformation if and only if it has no special adjoints. This, in our case, will translate into the first case of the theorem. Another interesting feature is that if $\kappa(A) = 0$ or 1, the divisor class group of A could be smaller than that of $\hat{A}$. For instance, if $\kappa(A) = 0$ and $\hat{A} \simeq K[[X^4, X^3Y, X^2Y^2, XY^3, Y^4]]$, then $Cl(A) \simeq \mathbb{Z}/2\mathbb{Z}$ and $Cl(\hat{A}) \simeq \mathbb{Z}/4\mathbb{Z}$. Also, if X is a projective normal surface and $x \in X$ such that $X - \{x\}$ is smooth, $\mathcal{O}_{X,x} = K[[X^n, X^{n-1}Y,\dots,Y^n]]$, then $\kappa(\mathcal{O}_{X,x}) = -\infty \Leftrightarrow X$ has $\mathbb{P}'$-ruling and $\kappa(\mathcal{O}_{X,x}) = 0$ or 1 implies, X has a pencil of elliptic curves. This permits us to compute the Chow group of such surfaces. So for instance, if $V \subset X$ is an affine open subset of X,X as above, then $\kappa(\mathcal{O}_{X,x}) = -\infty$ implies vector bundles over V splits as a trivial bundle and a line bundle.

§2. Rational double points:

In § 1, we saw that the uniqueness of local rings has something to do with its Kodaira dimension being $-\infty$. There is a very prominent class of local rings with Kodaira dimension $-\infty$, namely, rational double points. So let A be a normal two dimensional geometric local ring, which is rational. Assume that A has a rational double point. The completions of such rings are well classified [L]. They are classified as A_n-type, D_n-type and the exceptional ones. So it is natural to ask that if A and B are any two such local rings with $\hat{A} \simeq \hat{B}$, then is $A \simeq B$? The answer we have obtained is not complete, but suggests that barring some exceptions, it is so. So let me state the theorems:

Let $R = K[X,Y,Z]_{(X,Y,Z)}$.

Theorem 2:

Let $\hat{A} \simeq \hat{R}/(Z^{n+1} - XY)$. ($A_n$-type).

Then,

i) $A \simeq R/(Z^{n+1} - XY)$ if $n \neq 7,8$.

ii) if $n = 7$, $A \simeq R/(Z^8 - XY)$ or $R/(Z^2-(Y-X^2)(Y-X^2-Y^2))$

iii) if $n = 8$, $A \simeq R/(Z^9 - XY)$ or $R/(Z^2-(X+Y^3)^2 - X^3)$

Theorem 3:

i) If $\hat{A} \simeq \hat{R}/(X^4 + Y^3 + Z^2)$, then $A \simeq R/(X^4 + Y^3 + Z^2)$ – (E_6)

ii) If $\hat{A} \simeq \hat{R}/(Y^3 + X^3Y + Z^2)$, then $A \cong R/(Y^3+X^3Y + Z^2)$ – (E_7)

iii) If $\hat{A} \simeq \hat{R}/(X^5 + Y^3 + Z^2)$, then either $A \cong R/(X^5 + Y^3 + Z^2)$
or $A \cong R/(X^4Y + X^5 + Y^3 + Z^2)$ (E_8)

These theorems are proved by exhibiting a pencil of rational curves on the projective normal surface containing above singularities. The special cases in Theorem 2, ii) and iii) occur, when there is no such rational pencils. In the cases of E_6, E_7 and E_8 there are no rational pencils, but only elliptic pencils. In case the singularity is of type E_6 or E_7, one utilises these elliptic pencils to show uniqueness. But in case of E_8, one gets either an elliptic pencil with all smooth members having constant j-function zero or varying values of the j-function. These give two non-isomorphic E_8-singularities.

In case of D_n-type singularity we can prove the following:

Theorem 4:

Let X be a projective, normal rational surface and $x \in X$ a D_n-type singularity. Also assume that $X - \{x\}$ is smooth. Then, if $n \neq 8$, X has a pencil of rational curves. If $n = 8$, then X either has a pencil of rational curves or a pencil of elliptic curves.

In this case, I am unable to prove (or disprove) the uniqueness of local rings. As indicated in the previous section, one may use these results to compute Chow groups of certain normal rational surfaces. Also, as before, the 'special' local rings have smaller class groups than one would expect.

Finally a few words about the proofs. The concept of logarithmic Kodaira dimension was effectively used by Miyanishi and Sugie [M-S] in exhibiting $\mathbb{A}'$-pencils on surfaces. The techniques are modified to obtain $\mathbb{P}'$-pencils and the methods are very similar. One uses a theorem of Fujita [F] to find suitable exceptional curves to blow down and bring the surfaces to a minimal case, where one can study the rational pencil (if it exists) effectively. The details can be found in [MK] and [MK-MP].

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